

Motivic classes of varieties and stacks with applications to Higgs bundles

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Introduction

Our primary goal is to give an introduction to motivic classes of varieties and stacks with applications to moduli of Higgs bundles.

Sources of Motivation:

- The work of Hausel, Letellier, Rodriguez-Villegas, and others regarding mixed Hodge polynomials of character varieties.
- Point counting for algebraic varieties and stacks over a finite field \mathbb{F}_q . Specifically, computations for moduli stacks of Higgs bundles done by Mozgovoy, Schiffmann, and Mellit.

Counting points of varieties over a finite field

Example

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field \mathbb{F}_q with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathrm{GL}(n)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$ considered as the space of n linearly independent columns.

Motivic classes of varieties

Preliminary definition

For any field k , one defines the abelian group $\text{Mot}_{\text{var}}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:

$$[X] = [Y] + [X - Y] \text{ where } Y \text{ is a closed subvariety of } X.$$

The class $[X]$ in $\text{Mot}_{\text{var}}(k)$ is called the motivic class of the variety X . We can give a ring structure for $\text{Mot}_{\text{var}}(k)$ by $[X] \cdot [Y] := [(X \times_k Y)_{\text{red}}]$ with the unit element $[\text{Spec } k] = 1$.

Note that $[\emptyset] = 0$. Similarly we can define the motivic classes of schemes of finite type over k , by $[X] := [X_{\text{red}}]$.

Realizations of the motivic classes

Lemma

Let A be an abelian group and f be an A -valued function on isomorphism classes of k -varieties such that for all closed subvarieties $Y \subset X$, we have $f(X) = f(Y) + f(X - Y)$. Then there is a unique $\tilde{f} : \text{Mot}_{\text{var}}(k) \rightarrow A$ such that for all varieties X we have $\tilde{f}([X]) = f(X)$.

Example

- Point counting: For $k = \mathbb{F}_q$, define $\# : \text{Mot}_{\text{var}}(k) \rightarrow \mathbb{Z}$ as $\#[X] = |X(\mathbb{F}_q)|$, the number of rational points over \mathbb{F}_q .
- Euler characteristic: For $k = \mathbb{C}$, define $\chi : \text{Mot}_{\text{var}}(k) \rightarrow \mathbb{Z}$ such that $\chi([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q})$ is the Euler characteristic of X .
- E -polynomial: For $k = \mathbb{C}$, define $E : \text{Mot}_{\text{var}}(k) \rightarrow \mathbb{Z}[u, v]$ by setting $E([X]) = \sum_{k,p,q} (-1)^k h^{k,p,q} u^p v^q$ such that $h^{k,p,q} = \dim_{\mathbb{C}} H_c^{k,p,q}(X)$ for any quasi-projective variety X .

Examples

So, instead of performing computations of invariants directly, we can instead perform computations of motivic classes in $\text{Mot}_{\text{var}}(k)$. Letting $[\mathbb{A}_k^1] = \mathbb{L}$, we see that the same methods used for counting points over \mathbb{F}_q give us:

Example

- $[\mathbb{A}^n] = \mathbb{L}^n$.
- $[\mathbb{P}^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$.
- $[\text{GL}(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$.

Local zeta function

Definition

The local zeta function of a variety X is defined as

$$Z(X, t) = \exp \left(\sum_{k=1}^{\infty} \frac{|X(\mathbb{F}_{q^k})|}{k} t^k \right)$$

We prefer to rewrite local zeta function in terms of symmetric products.

Proposition

$$Z(X, t) = \sum_{n \geq 0} |\mathrm{Sym}^n X(\mathbb{F}_q)| \cdot t^n$$

where $\mathrm{Sym}^n X$ is the symmetric product X^n/S_n (with the convention that $\mathrm{Sym}^0 X = \mathrm{Spec} k$).

Compare the two versions of local zeta function

We verify the above proposition for the coefficient of t^2 .

Example

For the original local zeta function, we have two cases $2 = 1 + 1$ and $2 = 2$ as follows:

$$\frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

For the symmetric product $|Sym^2 X(\mathbb{F}_q)|$, we have the following two cases. If $x, y \in \mathbb{F}_q$, the coefficient is $|Sym^2(X(\mathbb{F}_q))|$. If $x \in \mathbb{F}_{q^2} - \mathbb{F}_q$, the possible type is $(x, \phi(x))$, where ϕ is a Frobenius morphism. Finally we have

$$|Sym^2(X(\mathbb{F}_q))| + \frac{|X(\mathbb{F}_{q^2})| - |X(\mathbb{F}_q)|}{2} = \frac{|X(\mathbb{F}_{q^2})|}{2} + \frac{(|X(\mathbb{F}_q)|)^2}{2}.$$

Motivic zeta function

Definition

We can define the motivic zeta function for every quasi-projective variety X :

$$Z_X(t) = \sum_{n \geq 0} [\text{Sym}^n X] \cdot t^n \in 1 + t \cdot \text{Mot}_{\text{var}}(k)[[t]].$$

- If $k = \mathbb{F}_q$, then the image of this series under $\# : \text{Mot}_{\text{var}}(k) \rightarrow \mathbb{Z}$ coincides with the local zeta function.

Radicial and surjective morphism

- The morphism $f : X \rightarrow Y$ is called radicial if f is injective, and for every $x \in X$, the extension of residue fields $k(f(x)) \hookrightarrow k(x)$ is purely inseparable.

Example

- (1) Every closed immersion is radicial. In particular, $\mathrm{Spec}(R/I) \rightarrow \mathrm{Spec} R$ is radicial.
- (2) For any variety X over a field of characteristic p , The absolute Frobenius morphism $F_X : X \rightarrow X$ is given by the identity on the underlying topological space with

$$F^\#_X : \mathcal{O}_X \rightarrow \mathcal{O}_X \text{ given by } g \mapsto g^p.$$

The absolute Frobenius morphism is radicial and surjective.

Radical and surjective relation

- To compute the motivic zeta function, we need to relate $Sym^n X$ with $Sym^n Z$ and $Sym^n(X - Z)$.

Let S_n act on a variety X and let Z be a S_n -invariant closed subvariety. The problem is that in characteristic $p > 0$, for the following morphism

$$\pi : X \rightarrow X/S_n$$

the image of Z in X/S_n is not in general isomorphic to Z/S_n , i.e.

$$\pi(Z) \not\cong Z/S_n.$$

However, there is a natural radical and surjective morphism between them.

Motivic classes of varieties

Definition

For any field k , one defines the abelian group $\text{Mot}_{\text{var}}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:

- 1) $[X] = [Y] + [X - Y]$ where Y is a closed subvariety of X ,
- 2) $[X] = [Y]$ if there is a radicial and surjective morphism $X \rightarrow Y$ of varieties over k .

The class $[X]$ in $\text{Mot}_{\text{var}}(k)$ is called the motivic class of the variety X . We can give a ring structure for $\text{Mot}_{\text{var}}(k)$ by $[X] \cdot [Y] := [(X \times_k Y)_{\text{red}}]$ with the unit element $[\text{Spec } k] = 1$.

Note that $[\emptyset] = 0$. Similarly we can define the motivic classes of schemes of finite type over k , by $[X] := [X_{\text{red}}]$.

Remarks

- In characteristic 0, $\text{Mot}_{\text{var}}(k)$ coincides with the the first definition. In fact, the second relation is always true because every radicial and surjective morphism becomes an isomorphism after passing to locally closed stratifications.

Proposition (Mustață for a) and Totaro for b))

If X is a quasi-projective variety, Y is a closed subvariety of X , then

a) $Z_X(t) = Z_Y(t)Z_{X-Y}(t)$,

b) $Z_{X \times \mathbb{A}_k^n}(t) = Z_X(\mathbb{L}^n t)$.

Counting points of projective spaces

We are interested in performing computations for quotients. One of the motivation comes from the following computations for projective spaces.

Example

- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathbb{P}^n|$ can also be computed by presenting \mathbb{P}^n as a quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathrm{GL}(1)$.

$$\left| \frac{\mathbb{A}^{n+1} - \{0\}}{\mathrm{GL}(1)} \right| = \frac{q^{n+1} - 1}{q - 1} = 1 + q + \cdots + q^n$$

Counting points of groupoids

If \mathcal{G} is a groupoid, we can define the volume of \mathcal{G} as a weighted sum:

$$|\mathcal{G}| := \sum_{G \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(G)|}.$$

Example (Funny)

The volume of the groupoid of finite sets is e . Indeed, for any finite set of size n , the automorphism group is of order $n!$:

$$|\mathcal{G}| = \sum_{G \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(G)|} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Counting points of algebraic stacks over a finite field

- We can try to do similar computations even when the group action is not free and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If \mathcal{Y} is an algebraic stack over \mathbb{F}_q , we can generalize point counting for varieties by defining the volume (also known as the mass) of \mathcal{Y} as a weighted sum over all its \mathbb{F}_q -rational points:

$$|\mathcal{Y}| := \sum_{y \in \text{Ob}(\mathcal{Y}(\mathbb{F}_q))} \frac{1}{|\text{Aut}(y)|}.$$

This is not guaranteed to converge. However, all the stacks we consider are of finite type, for which it converges (the sum is finite).

Example: volume of the stack of vector bundles

We give an example where the stack is not of finite type but the volume converges.

Siegel formula

Let X be a smooth projective curve of genus g over \mathbb{F}_q . Let $\mathcal{B}un_{r,d}(X)$ be the moduli stack parametrizing isomorphism classes of rank r , degree d vector bundles over X . The volume over \mathbb{F}_q may be computed as:

$$|\mathcal{B}un_{r,d}(X)| = \frac{q^{(r^2-1)(g-1)}}{q-1} |\text{Jac}(X)| \prod_{i=2}^r Z_X(q^{-i})$$

where $\text{Jac}(X)$ is the Jacobian of X , i.e. the moduli space of degree 0 line bundles on X .

Motivic classes of stacks

We use the convention that all the stacks considered will be Artin stacks locally of finite type over a field k such that the stabilizers of points are affine.

Definition

One defines the abelian group $\text{Mot}(k)$ as the group generated by isomorphism classes of stacks of finite type over k modulo the following relations:

- 1) $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} - \mathcal{Y}]$ where \mathcal{Y} is a closed substack of \mathcal{X} ,
- 2) $[\mathcal{X}] = [\mathcal{Y}]$ if there is a radicial and surjective morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of stacks over k .
- 3) $[\mathcal{X}] = [\mathcal{Y} \times \mathbb{A}_k^r]$ where $\mathcal{X} \rightarrow \mathcal{Y}$ is a vector bundle of rank r .

The class $[\mathcal{X}]$ in $\text{Mot}(k)$ is called the motivic class of the stack \mathcal{X} .

Similarly we can give a ring structure for $\text{Mot}(k)$ by $[\mathcal{X}] \cdot [\mathcal{Y}] := [\mathcal{X} \times_k \mathcal{Y}]$.

Remarks

- For the motivic classes of varieties, the third relation is trivial since every vector bundle over a variety is locally trivial in the Zariski topology. However in motivic classes of stacks, $\{pt\}/GL(n) = BGL(n)$, so we have a vector bundle

$$V := \mathbb{A}_k^n \times^{GL(n)} \{pt\} \rightarrow BGL(n),$$

but it is not Zariski locally trivial.

- The morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is radicial if for any morphism $\text{Spec}(K) \rightarrow \mathcal{Y}$ where K is a field, the reduction $(\text{Spec}(K) \times_{\mathcal{Y}} \mathcal{X})_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K .

Quotient stacks and stratifications

Definition

Let X be a quasi-projective variety and let G be a linear algebraic group acting on X . We define the quotient stack X/G as

$$X/G : (\text{Sch}/k)^{op} \rightarrow \text{Grpds}$$

$$S \mapsto \{(E, \alpha) : E \xrightarrow{G} S \text{ is a } G\text{-torsor and} \\ \alpha : E \rightarrow X \text{ is a } G\text{-equivariant morphism}\} \\ + \{\text{isomorphism of pairs}\}$$

Definition

We define the stratification of a stack \mathcal{X} as a decomposition $\mathcal{X} = \coprod_i \mathcal{X}_i$ into locally closed substacks \mathcal{X}_i with a partial ordering I such that

$$\overline{\mathcal{X}_j} \subset \bigcup_{i \leq j} \mathcal{X}_i.$$

Affine stabilizer condition

We need the affine stabilizer condition since every such stack has a stratification by global quotients of the form $X/\mathrm{GL}(n)$ with X a variety.

Proposition (Kresch, 1999)

- (1) Let \mathcal{X} be a stack. Then \mathcal{X} admits a stratification by quotient stacks if and only if for every geometric point the stabilizer group is affine.
- (2) \mathcal{X} is a quotient stack if and only if there exists a variety X with the action of $\mathrm{GL}(n)$ such that

$$\mathcal{X} \cong X/\mathrm{GL}(n).$$

Relation between the motivic classes

In special cases, we have the following computations.

Lemma (Ekedahl, 2009)

In $\text{Mot}(k)$ we have the following results.

- 1) $[GL(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$,
- 2) For a stack \mathcal{X} , we have $[\mathcal{X}/GL(n)] = [\mathcal{X}]/[GL(n)]$,
- 3) $[*/GL(n)] = 1/((\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1}))$.

With the above lemma, we find a relation between the motivic classes.

Theorem (Ekedahl, 2009 and L, 2024)

There is a natural group isomorphism :

$$\text{Mot}_{\text{var}}(k)[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1} | i > 0] \cong \text{Mot}(k)$$

Completion of the motivic classes

- Let $F^m \text{Mot}(k)$ be the subgroup generated by the classes of stacks of dimension $\leq -m$. This is a ring filtration and we define the completed ring $\overline{\text{Mot}}(k)$ as the completion of $\text{Mot}(k)$ with respect to this filtration.

The case of SL_n -torsor

Let X be a smooth geometrically connected projective curve of genus g over k . Let $\mathcal{B}un_{\text{SL}_n}$ be the moduli stack parametrizing isomorphism classes of SL_n -torsor over X . The motivic classes may be computed as:

$$[\mathcal{B}un_{\text{SL}_n}] = \mathbb{L}^{(n^2-1)(g-1)} \prod_{i=2}^n Z(X, \mathbb{L}^{-i}).$$

Motivic zeta function and plethystic exponents

The following theorem generalizes the motivic zeta function.

Theorem (Fedorov-Soibelman-Soibelman, 2018 and L, 2024)

One can uniquely extend the assignment X to Z_X to a continuous homomorphism of topological groups

$$Z : \overline{\text{Mot}}(k) \rightarrow (1 + z\overline{\text{Mot}}(k)[[z]])^\times$$

such that for any $A \in \overline{\text{Mot}}(k)$ and any $n \in \mathbb{Z}$ we have $Z_{\mathbb{L}^n A}(z) = Z_A(\mathbb{L}^n z)$. More precisely, any class $A \in \overline{\text{Mot}}(k)$ can be written as the limit of a sequence $([Y_i] - [Z_i])/\mathbb{L}^{n_i}$, where Y_i and Z_i are varieties. Thus we define

$$Z_A(z) = \lim_{i \rightarrow \infty} \frac{Z_{Y_i}(\mathbb{L}^{-n_i} z)}{Z_{Z_i}(\mathbb{L}^{-n_i} z)}.$$

Plethystic exponents

Definition

Now we can consider the ring of formal power series in two variables $\overline{\text{Mot}}(k)[[z, w]]$. Let $\overline{\text{Mot}}(k)[[z, w]]^+$ denote the ideal of power series with vanishing constant term and let $(1 + \overline{\text{Mot}}(k)[[z, w]]^+)^{\times}$ be the multiplicative group of series with constant term equal 1. We can define the plethystic exponent $\text{Exp} : \overline{\text{Mot}}(k)[[z, w]]^+ \rightarrow (1 + \overline{\text{Mot}}(k)[[z, w]]^+)^{\times}$ by

$$\text{Exp} \left(\sum_{r,d} A_{r,d} w^r z^d \right) = \prod_{r,d} Z_{A_{r,d}}(w^r z^d).$$

Exp is an isomorphism of abelian groups, so we can define the plethystic logarithm Log as its inverse.

Stacks of Higgs bundles

Fix a smooth projective geometrically connected curve X over k .

Definition

A Higgs bundle on X is a pair (E, Φ) where E is a vector bundle on X and $\Phi : E \rightarrow E \otimes \Omega_X$ is an \mathcal{O}_X -linear morphism from E to E "twisted" by the sheaf of differential 1-forms Ω_X . The rank of the pair (E, Φ) is the rank of E , similarly the degree is the degree of E .

Definition

We denote by $\mathcal{Higgs}_{r,d}$ the moduli stack of rank r degree d Higgs bundles on X . This is an Artin stack locally of finite type over k .

Note that the motivic classes of $\mathcal{Bun}_{r,d}$ converge, but the motivic classes of $\mathcal{Higgs}_{r,d}$ diverge, that is, have infinite volumes.

Moduli stacks of semistable Higgs bundles

Definition

The Higgs bundle (E, Φ) is called semistable if for any subbundle $F \subset E$ preserved by Φ ,

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}.$$

- This is an open condition compatible with field extensions, so we use $\mathcal{Higgs}_{r,d}^{SS} \subset \mathcal{Higgs}_{r,d}$ to denote the open substack of semistable Higgs bundles.
- $\mathcal{Higgs}_{r,d}^{SS}$ is a stack of finite type, thus the corresponding motivic classes converge.

Zeta functions and partitions

Definition

We define “normalized” zeta-function by $\tilde{Z}_X(z) := z^{1-g} Z_X(z)$ and the “regularized” zeta-function by setting

$$Z_X^*(\mathbb{L}^{-u} z^v) = \begin{cases} Z_X^*(\mathbb{L}^{-u} z^v) & \text{if } v > 0 \text{ or } u > 1, \\ \operatorname{res}_{z=\mathbb{L}^{-1}} Z_X(z) \frac{dz}{z} & \text{if } (u, v) = (1, 0), \\ \operatorname{res}_{z=1} Z_X(z) \frac{dz}{z} & \text{if } (u, v) = (0, 0). \end{cases}$$

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0)$ be a partition. This can also be written as $\lambda = 1^{r_1} 2^{r_2} \dots t^{r_t}$, where r_i denotes the number of occurrences of i among λ_j , $1 \leq j \leq l$. The Young diagram of λ is the set of the points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq i \leq \lambda_j$. For a box $s \in \lambda$, the arm $a(s)$ (resp. leg $l(s)$) is the number of boxes lying strictly to the right of (resp. strictly above) s .

Explicit formulae I

For a partition λ , set

$$J_{\lambda}^{mot}(z) = \prod_{s \in \lambda} Z_X^*(\mathbb{L}^{-1-l(s)} z^{a(s)}) \in \overline{\text{Mot}}(k)[[z]],$$

where the product is over all boxes of the Young diagram corresponding to the partition. In particular for the empty Young diagram λ we have

$$J_{\lambda}^{mot}(z) = 1.$$

Set

$$L^{mot}(z_n, \dots, z_1) = \frac{1}{\prod_{i < j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right)} \times \\ \times \sum_{\sigma \in S_n} \sigma \left\{ \prod_{i < j} \tilde{Z}_X\left(\frac{z_i}{z_j}\right) \frac{1}{\prod_{i < n} \left(1 - \mathbb{L} \frac{z_{i+1}}{z_i}\right)} \cdot \frac{1}{1 - z_1} \right\}.$$

Residue

Assume that $R(z)$ is a rational function with coefficients in $\overline{\text{Mot}}(k)$, that is, an element of the total ring of fractions of $\overline{\text{Mot}}[z]$. We say that it has at most first order pole at $x \in \overline{\text{Mot}}$ if it can be written as

$$\frac{P(z)}{(x-z)Q(z)},$$

where $Q(x)$ is not a zero divisor in $\overline{\text{Mot}}(k)$. In this case we can define

$$\text{res}_{z=x} R(z) dz = \frac{P(x)}{Q(x)}.$$

On the other hand, if $Q(x)$ is invertible, we can expand $R(z)$ in powers of z . The residue of the corresponding series may or may not exist, but if it exists, it is equal to the residue of the rational function.

Explicit formulae II

For a partition $\lambda = 1^{r_1} 2^{r_2} \dots t^{r_t}$ such that $\sum_i r_i = n$, set $r_{<i} = \sum_{k<i} r_k$ and denote by res_λ the iterated residue along

$$\begin{array}{ccc} \frac{z_n}{z_{n-1}} = \mathbb{L}^{-1}, & \frac{z_{n-1}}{z_{n-2}} = \mathbb{L}^{-1}, & \dots, & \frac{z_{2+r_{<t}}}{z_{1+r_{<t}}} = \mathbb{L}^{-1}, \\ \vdots & \vdots & & \vdots \\ \frac{z_{r_1}}{z_{r_1-1}} = \mathbb{L}^{-1}, & \frac{z_{r_1-1}}{z_{r_1-2}} = \mathbb{L}^{-1}, & \dots, & \frac{z_2}{z_1} = \mathbb{L}^{-1}. \end{array}$$

Set

$$\tilde{H}_\lambda^{\text{mot}}(z_{1+r_{<t}}, \dots, z_{1+r_{<i}}, \dots, z_1) := \text{res}_\lambda \left[L^{\text{mot}}(z_n, \dots, z_1) \prod_{\substack{j=1 \\ j \notin \{r_{<i}\}}}^n \frac{dz_j}{z_j} \right]$$

and

$$H_\lambda^{\text{mot}}(z) := \tilde{H}_\lambda^{\text{mot}}(z^t \mathbb{L}^{-r_{<t}}, \dots, z^i \mathbb{L}^{-r_{<i}}, \dots, z).$$

Formulae for semistable Higgs bundles

Theorem (Fedorov-Soibelman-Soibelman, 2018 and L, 2024)

For sufficiently large e , we have $[\mathcal{Higgs}_{r,d}^{ss}] = H_{r,d+er}$, where $H_{r,d}$ is defined via the generating function

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H_{r,d} w^r z^d = \text{Exp} \left(\sum_{d/r=\tau} B_{r,d} w^r z^d \right),$$

where τ is any rational number. We define $B_{r,d}$ by

$$\sum_{\substack{r,d \in \mathbb{Z}_{\geq 0} \\ (r,d) \neq (0,0)}} B_{r,d} w^r z^d = \mathbb{L} \text{Log} \left(\sum_{\lambda} \mathbb{L}^{(g-1)\langle \lambda, \lambda \rangle} J_{\lambda}^{\text{mot}}(z) H_{\lambda}^{\text{mot}}(z) w^{|\lambda|} \right).$$

The sum is over all partitions and $\langle \lambda, \lambda \rangle = \sum_i (\lambda'_i)^2$.

Thank you!